Math3506, 2012. Model solutions

Qu1

(a) This is a predator-prey model. There are intraspecifc competition terms $-bx^2, -fy^2$. ex is the predator per capita growth from consuming prey x and $-cy$ is the per capita reduction in prey x due to consumption by predator y. The carrying capacity for x is a/b and for y is 0.

(b) Steady states are $(0,0)$, $(a/b, 0)$ and any interior steady state solves

$$a - bx - cy = 0, \quad -d + ex - fy = 0.$$

Solving we get

$$(x^*, y^*) = \frac{1}{bf + ce}(fa + cd, ea - bd),$$

so we require $ae > bd$ for an interior steady state to exist.

For stability, we find the stability matrix

$$M = \begin{pmatrix} (a - bx - cy) - bx & -cx \\ ey & (-d + ex - fy) - fy \end{pmatrix}.$$

At $(0,0)$ we have

$$M = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix},$$

which has eigenvalues $a, -d$ hence $(0,0)$ is a saddle. At $(a/b, 0)$ we have

$$M = \begin{pmatrix} -a & -ac/b \\ 0 & -d + ea/b \end{pmatrix},$$

which has eigenvalues which are opposite sign and hence $(a/b, 0)$ is a saddle. At the interior steady state (x^*, y^*) we have

$$M = \begin{pmatrix} -bx^* & -cx^* \\ ey^* & -fy^* \end{pmatrix}.$$

Thus $\lambda_1 + \lambda_2 = -bx^* - fy^* < 0$ and $\lambda_1\lambda_2 = x^*y^*(bf + ce) > 0$ when the interior steady state exists. Hence both eigenvalues have negative real part and the interior steady state is locally stable when it exists.

(c) (i) $x(0) = 0, y(0) > 0$. $x(t) = 0\ \forall t$ and $\dot{y} = y(-d - fy)$ so $x(t) = 0, \lim_{t\to\infty} y(t) = 0$. Similarly if (ii) $y(0) = 0$ then $y(t) = 0\ \forall t$, $\dot{x} = x(a - bx)$ and if $x(0) > 0$ then $x(t) \to a/b$ monotonically, i.e. $\lim_{t\to\infty} x(t) = a/b, y(t) = 0$.

(d) See fig 1

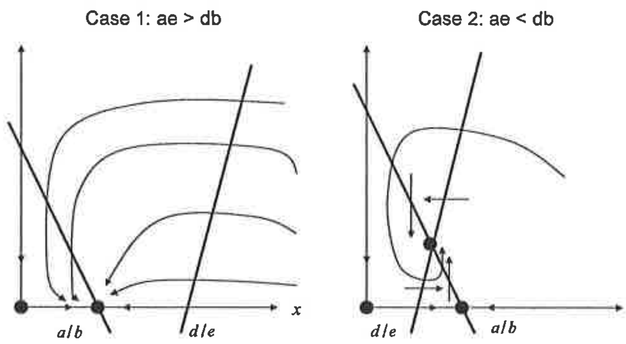

Figure 1: Qu 1 part . Left $ae > bd$, right $ae \leq bd$.

Qu2

(a) Steady states are solutions of $N = rN/(1+N^\alpha)$, so we have $N = 0$ and $N = (r-1)^{1/\alpha}$ which exists if $r > 1$.
Stability is determined by the eigenvalues $\lambda = f'(N^*)$ where N^* is the steady state. But $f'(N) = \frac{r(K-(\alpha-1)N^\alpha)}{(K+N^\alpha)^2}$. Hence $\lambda = f'(0) = r$ so the origin is stable if $r < 1$. We need $r > 1$ for the non-zero steady state to exist. $f'((r-1)^{1/\alpha}) = \frac{1-(\alpha-1)(r-1)}{r}$. Then $(r-1)^{1/\alpha}$ is stable for $1 < r < \frac{\alpha}{\alpha-2}$.

(b) See fig 2

(c) Solve $f^2(N) = N$:

$$\frac{r\frac{rN}{1+N^\alpha}}{1+\left(\frac{rN}{1+N^\alpha}\right)^\alpha} = N$$

Remove case $N = 0$ and set $x = 1 + N^\alpha$ to obtain

$$x^\alpha - r^2 x^{\alpha-1} + r^\alpha x - r^\alpha = 0.$$

$x = r$ is one root corresponding to the steady state $N = (r-1)^\alpha$. Since $\alpha \leq 3$ there can be at most two other real roots x_1, x_2 and they must exceed one for the corresponding N_1, N_2 to be positive.

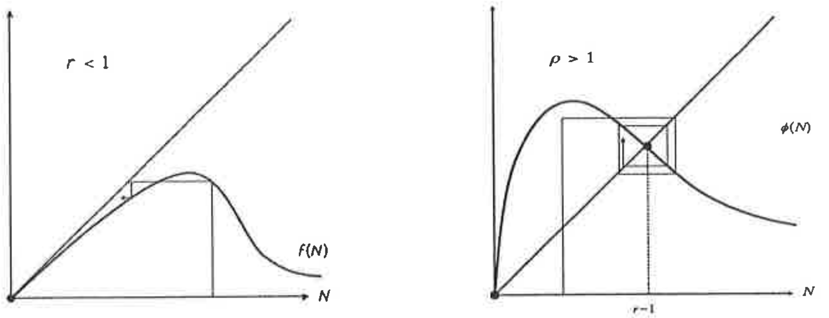

Figure 2: Qu2 part . Left $0 < r < 1$ and right $1 < r$.

(d) For $\alpha < 3$, a 2-cycle is not possible by the previous part. if $\alpha = 3$ then we have $x^3 - r^2x^2 + r^3x - r^3 = (x - r)(x^2 - (r^2 - r)x + r^2)$. Hence we need $(r^2 - r)^2 > 4r^2$ for real x which translates to $r > 3$. Moreover $(r^2 - r)x = x^2 + r^2 > r^2$ so each $x > 1$.

Qu3

(a) Both intra and interspecific competition.

(b) $\begin{pmatrix} e & b \\ d & f \end{pmatrix} (x, y)^T = (a, c)^T$ so $(x, y)^T = \frac{1}{ef - bd} \begin{pmatrix} f & -b \\ -d & e \end{pmatrix} (a, c)^T = \left(\frac{fa - bc}{ef - bd}, \frac{ec - ad}{ef - bd} \right)$ so we need $f/b > c/a > d/e$ or $f/b < c/a < d/e$.

(c)

$$
\begin{aligned}
\dot{V} &= d(\dot{x} - x^* \frac{\dot{x}}{x}) + b(\dot{y} - y^* \frac{\dot{y}}{y}) \\
&= d\frac{\dot{x}}{x}(x - x^*) + b\frac{\dot{y}}{y}(y - y^*) \\
&= d(a - ex - by)(x - x^*) + b(c - dx - fy)(y - y^*) \\
&= -d(e(x - x^*) + b(y - y^*))(x - x^*) - b(d(x - x^*) + f(y - y^*))(y - y^*) \\
&= -deX^2 - 2bdXY - bfY^2
\end{aligned}
$$

where $X = x - x^*, Y = y - y^*$. Now $(2bd)^2 - 4(de)(bf) = 4bd(bd - ef) < 0$ so that $-deX^2 - 2bdXY - bfY^2$ has no real roots and hence is negative unless $X = Y = 0$. By the Lyapunov theorem, $(x(t), y(t)) \to (x^*, y^*)$ for all $(x(0), y(0))$ in the interior of the first quadrant.

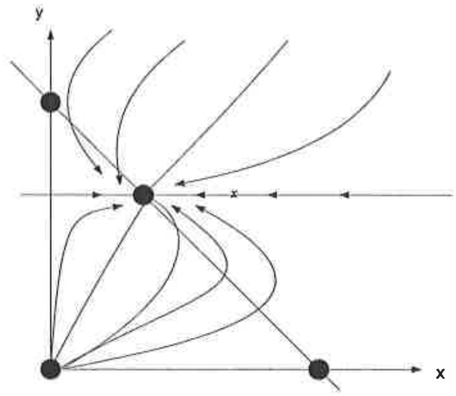

Figure 3: Qu3 part

Qu4

(a)

$$L = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & p_0 b \\ p_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-1} & 0 \end{pmatrix}.$$

(b) Characteristic polynomial reads $c(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} p_0 p_1 \cdots p_{n-1} b$, so that the eigenvalues are $\lambda_k = re^{2k\pi i/n}$ for $k = 0, 1, \ldots, n-1$ where $r = (p_0 p_1 \cdots p_{n-1} b)^{1/n}$.

(c) Since the eigenvalues are distinct, we have a linearly dependent set of eigenvectors $v_0, \ldots, v_{n-1}$ and any initial population has an expansion $N(0) = \Re\{\sum_{k=0}^{n-1} \alpha_k v_k\}$. Thus

$$L^t N(0) = \Re\{\sum_{k=0}^{n-1} \lambda_k^t \alpha_k v_k\} = \Re\{\sum_{k=0}^{n-1} r^t e^{2k\pi ti/n} \alpha_k v_k\}.$$

When we look at the age distribution the r^t cancels and we are left with $\mathbf{X}$ as the ratio of two periodic functions of period n, i.e. a periodic function of period n.

(d) The population will die out if $r < 1$.

Qu5

(a) $\rho(t)$ is the time-dependent intrinsic growth rate of the population, and $K(t)$ is the time-dependent carrying capacity.

(b) Write $M(t) = N(t)\exp(-\int_0^t \rho(s)\,ds)$. Then $M_0 = M(0) = N(0) = N_0$, and

$$\begin{aligned}\frac{dM}{dt} &= \left[\frac{dN}{dt} - \rho(t)N(t)\right]\exp(-\int_0^t \rho(s)\,ds) \\ &= \left[\rho(t)N\left(1-\frac{N}{K(t)}\right) - \rho(t)N(t)\right]\exp(-\int_0^t \rho(s)\,ds) \\ &= -H(t)M(t)^2 \text{ where } H(t) = \frac{\rho(t)}{K(t)}\exp\left(\int_0^t \rho(s)\,ds\right).\end{aligned}$$

Thus $dM/M^2 = -H(t)dt$ and integrating yields

$$M(t) = \frac{1}{\left\{\frac{1}{M_0} + \int_0^t H(u)\,du\right\}} = \frac{M_0}{1 + M_0\int_0^t H(u)\,du}.$$

Finally, in terms of N we have

$$\begin{aligned}N(t) &= \frac{N_0 \exp\left(\int_0^t \rho(s)\,ds\right)}{1 + N_0\int_0^t H(u)\,du}, \\ H(u) &= \frac{\rho(u)}{K(u)}\exp\left(\int_0^u \rho(s)\,ds\right).\end{aligned}$$

(c) When $\rho(t) = r + \alpha\cos(2\pi t/T)$, we have

$$\int_0^u \rho(s)\,ds = ru + \frac{T\alpha}{2\pi}\sin(2\pi u/T).$$

Hence

$$H(u) = \frac{1}{\kappa}\frac{d}{du}\exp\left(ru + \frac{T\alpha}{2\pi}\sin(2\pi u/T)\right).$$

and

$$\int_0^t H(u)\,du = \frac{1}{\kappa}\left(\exp\left(rt + \frac{T\alpha}{2\pi}\sin(2\pi t/T)\right) - 1\right).$$

This gives

$$N(t) = \frac{N_0\exp\left(rt + \frac{T\alpha}{2\pi}\sin(2\pi t/T)\right)}{1 + N_0\frac{1}{\kappa}\left(\exp\left(rt + \frac{T\alpha}{2\pi}\sin(2\pi t/T)\right) - 1\right)}.$$

(d) If $r < 0$ then $N(t) \to 0$. If $r > 0$ then $N(t) \to \kappa$. If $r = 0$ then $N(t)$ is periodic, period T.